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1993 J. Phys. A: Math. Gen. 26 6033

(http://iopscience.iop.org/0305-4470/26/21/040)

J. Phys. A: Math. Gen. 26 (1993) 6033–6038. Printed in the UK

Time dependence and intrinsic irreversibility of the Pietenpol model

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Received 25 September 1992, in final form 16 August 1993

Abstract. We study a model for an unstable system for which the unperturbed Hamiltonian has a possibly infinite sequence of discrete states embedded in a continuous spectrum on $(-\infty, \infty)$. The perturbation has matrix elements only between a non-degenerate continuum and the eigenfunctions associated with the discrete spectrum. This idealization of the Stark effect has the soluble structure of the Friedrichs model. We show that the time dependence of the decay is a sum of exponential contributions plus a background contribution that may be arbitrarily small for any positive t. We discuss the structure of the generalized eigenstates in the Gel'fand triple associated with the resonances.

We study a system for which the unperturbed Hamiltonian has an absolutely continuous non-degenerate spectrum in $(-\infty, \infty)$ and a possibly infinite number of discrete states embedded in the continuum with eigenvalues m_k and eigenstates ϕ_k. The perturbation is defined by (we use $|f)$ for normalized states of the Hilbert space, and $|\lambda\rangle$, for example, for generalized states or, equivalently, spectral representation)

$$V = \sum_{k=1}^{N} \int_{-\infty}^{\infty} d\lambda \{ g_k(\lambda) |\lambda\rangle\langle\phi_k| + g_k^*(\lambda)|\phi_k\rangle\langle\lambda| \} \tag{1}$$

where N may be infinite. The form factors are taken to be continuous and such that (for f in the domain of V)

$$\|Vf\|^2 = \int_{-\infty}^{\infty} d\lambda |F(\lambda)|^2 + \sum_k |F_k|^2 < \infty \tag{2}$$

where

$$F(\lambda) = \sum_k g_k(\lambda)(\phi_k, f) \qquad F_k = \int_{-\infty}^{\infty} d\lambda g_k^*(\lambda)(\lambda|f). \tag{3}$$

Note that

$$g_k(\lambda) = \langle\lambda|V|\phi_k\rangle. \tag{4}$$

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This generalization, which we shall call the Pietenpol model [1], of the Friedrichs model [2] to a continuous spectrum in $(-\infty, \infty)$ is an idealization of the Stark system, where the potential of the unperturbed problem generates N bound states, and the perturbing potential is of rank N. In the case $N = \infty$, one may model a local potential. The result is a system of (lower half plane) complex poles of the resolvent and a total Hamiltonian with no real bound states. We show that for the N-Pietenpol system, for sufficient analyticity of the functions $g_k(\lambda)$ (and, for this study, the assumption of simple poles for the resolvent in its extension to the lower half plane), the time dependence of the decay law is that of a sum of exponential decay terms for any $t > 0$ to accuracy $\mathrm{O}(\mathrm{e}^{-td})$, where d is bounded by the domain of analyticity of $g_k(\lambda)$ in the lower half plane. For the corresponding model with semibounded continuous spectrum [2], exponential decay cannot be achieved for $t \to \infty$ [3].

Let us consider the amplitude

$$A_{ij}(t) = (\phi_i|\mathrm{e}^{-\mathrm{i}Ht}|\phi_j) = \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}z\, R_{ij}(z)\mathrm{e}^{-\mathrm{i}zt} \tag{5}$$

where

$$R_{ij}(z) = \left(\phi_i \left|\frac{1}{z-H}\right| \phi_j\right) \tag{6}$$

for $t \geqslant 0$ and z in the upper half plane; the contour C runs from ∞ to $-\infty$ above the real axis. Then,

$$p_j(t) = \sum_{i=1}^{N} |A_{ij}(t)|^2 \tag{7}$$

is the probability that the initial state j remains among the states of the discrete subspace, and $1 - p_j(t)$ is the probability that this initial state decays to the continuum. With the form (1) for the potential, the second resolvent equation

$$\frac{1}{z-H} = \frac{1}{z-H_0} + \frac{1}{z-H_0}V\frac{1}{z-H} \tag{8}$$

implies the relations

$$R_{ij}(z) = \frac{1}{z-m_i}\delta_{ij} + \frac{1}{z-m_i}\int_{-\infty}^{\infty}\mathrm{d}\lambda\, g_i^*(\lambda)\left\langle\lambda\left|\frac{1}{z-H}\right|\phi_j\right) \tag{9}$$

and

$$\left\langle\lambda\left|\frac{1}{z-H}\right|\phi_j\right) = \frac{1}{z-\lambda}\sum_k g_k(\lambda)R_{kj}(z). \tag{10}$$

Substituting (10) into (9), we find the relation

$$\sum_k h_{ik}(z)R_{kj}(z) = \delta_{ij} \tag{11}$$

where

$$h_{ij}(z) = (z-m_i)\delta_{ij} - \int_{-\infty}^{\infty}\mathrm{d}\lambda\frac{g_i(\lambda)^* g_j(\lambda)}{z-\lambda}. \tag{12}$$

The determinant of $h_{ij}(z)$ can have no zeros in the upper half plane, since $R_{ij}(z)$ is analytic. The proof is direct in the rank one case $(h_{ij}(z) \to h(z))$ for which

$$\operatorname{Im} h(z) = \operatorname{Im} z \left\{ 1 + \int_{-\infty}^{\infty} \frac{|g(\lambda)|^2 \mathrm{d}\lambda}{|z-\lambda|^2} \right\}. \tag{13}$$

In the limit $z \to x = \mathrm{i}0$,

$$h(z) \to (x-m) + \mathrm{i}\pi |g(x)|^2 - P \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{|g(\lambda)|^2}{x-\lambda} \tag{14}$$

so that there is no zero on the real axis as well. In the general case, on the real axis, for $z \to x + \mathrm{i}0$,

$$h_{ij}(x) = (x-m_i)\delta_{ij} - P \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{g_i(\lambda)^* g_j(\lambda)}{x-\lambda} + \mathrm{i}\pi g_i(x)^* g_j(x). \tag{15}$$

For $|x - m_i| = \mathrm{O}(1)$, i.e. not $\mathrm{O}(g^2)$, $g_i(\lambda) = \mathrm{O}(g)$, we have, to $\mathrm{O}(g^2)$, for finite N,

$$\det(h_{ij}(x)) \cong \sum_j \Pi_{i\neq j}(x-m_i) \left\{ \mathrm{i}\pi |g_j(x)|^2 - P \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{|g_j(\lambda)|^2}{x-\lambda} \right\} \tag{16}$$

and there are, therefore, no zeros in the regions $(x - m_i)$ not $\mathrm{O}(g^2)$ on the real axis as well. If $x = m_{i^*} + \mathrm{O}(g^2)$ for some $i = i^*$,

$$\det(h_{ij}(x)) \cong \Pi_{i\neq i^*}(x-m_i) \left\{ x - m_{i^*} + \mathrm{i}\pi |g_{i^*}(x)|^2 - P \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{|g_{i^*}(\lambda)|^2}{x-\lambda} \right\}. \tag{17}$$

We therefore conclude, as for the usual Stark effect [4], that there are no discrete eigenvalues on the real axis (finite N).

Let us now analytically continue $h_{ij}(z)$ in the following way. We assume that the set of functions

$$W_{ij}(\lambda) = g_i^*(\lambda) g_j(\lambda) \tag{18}$$

is the boundary value on the real axis of a set of functions analytic in some domain S_W in the lower half plane. Then,

$$h'_{ij}(z) = (z-m_i)\delta_{ij} - \int_{-\infty}^{\infty} \frac{W_{ij}(\lambda)}{z-\lambda} \mathrm{d}\lambda + 2\pi \mathrm{i} W_{ij}(z) \tag{19}$$

has the value $h_{ij}(x)$ given in (15) in the limit $z \to x - \mathrm{i}0$. Hence there is a unique function

$$\overline{h}_{ij}(z) = \begin{cases} h_{ij}(z) & z \in uhp \\ h'_{ij}(z) & z \in S_W \end{cases} \tag{20}$$

which is analytic in the whole region $uhp \oplus S_W$, and is the analytic continuation of $h_{ij}(z)$ in S_W. If $|z - m_{i^*}| = \mathrm{O}(g^2)$, for finite N,

$$\det(h'_{ij}(z)) = \Pi_{i\neq i^*}(z-m_i) \left\{ z - m_i^* + 2\pi \mathrm{i} W_{i^*i^*}(z) - \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{|g_{i^*}(\lambda)|^2}{z-\lambda} \right\} + \mathrm{O}(g^4). \tag{21}$$

The imaginary part of the expression in brackets is

$$\operatorname{Im} z \left\{ 1 + \int_{-\infty}^{\infty} \frac{|g_{i^*}(\lambda)|^2}{|z-\lambda|^2} \right\} + 2\pi \operatorname{Re} W_{i^*i^*}(z) \tag{22}$$

which can clearly vanish for $\mathrm{Im}\, z < 0$. We may estimate the value of the root by recognizing, in (22), an approximation to the distribution

$$\lim_{\epsilon\to 0_-} \frac{\epsilon}{x^2+\epsilon^2} = -\pi\delta(x) \tag{23}$$

where $\mathrm{Im}\, z$ corresponds to ϵ. The integral then has the approximate value $-\pi|g_{i^*}(\mathrm{Re}\, z)|^2/\mathrm{Im}\, z$; since $\mathrm{Re}\, z = m_{i^*} + \mathrm{O}(g^2)$, we may also approximate $W_{i^*i^*}(z)$ by its real value from (18); the vanishing of the imaginary part of (22) then implies that, to order g^2,

$$\mathrm{Im}\, z \simeq -\pi|g_{i^*}(m_{i^*})|^2 \tag{24}$$

at the position of the zero. The corresponding real part of the root of the expression in curly brackets in (21) is found from

$$x = m_{i^*} + 2\pi\, \mathrm{Im}\, W_{i^*i^*}(x + \mathrm{i}\, \mathrm{Im}\, z) + \int_{-\infty}^{\infty} \frac{(x-\lambda)|g_{i^*}(\lambda)|^2}{(x-\lambda)^2 + (\mathrm{Im}\, z)^2} \mathrm{d}\lambda. \tag{25}$$

For $\mathrm{Im}\, z$ small, one can neglect $\mathrm{Im}\, W_{i^*i^*}(x + \mathrm{i}\,\mathrm{Im}\, z)$; the third term is approximately a principal part.

Let us consider the generalized eigenvalue problem [5]

$$Hf(z) = zf(z). \tag{26}$$

From the structure of the model (1), it follows that

$$m_i(\phi_i|f(z)) + \int_{-\infty}^{\infty} g_i^*(\lambda)(\lambda|f(z))\, \mathrm{d}\lambda = z(\phi_i|f(z))$$

and

$$z(\lambda|f(z)) = \lambda(\lambda|f(z)) + \sum_k g_k(\lambda)(\phi_k|f(z)). \tag{27}$$

Then, as for the semibounded case [5],

$$(\lambda|f(z)) = \frac{1}{z-\lambda}\sum_k g_k(\lambda)(\phi_k|f(z)) \tag{28}$$

defined for z in the upper half plane (as in our treatment of the resolvent). We obtain the eigenvalue equation

$$\sum_j h_{ij}(z)(\phi_j|f(z)) = 0. \tag{29}$$

We see from this expression that, as we have shown above, there is no solution in the upper half plane or on the real axis; the analytic continuation of the relation (for $\chi \in D$, a subset of the Hilbert space for which $\chi(\lambda) = (\lambda|\chi)$ is the boundary value of a function analytic in a domain of the lower half plane which contains the zeros of the determinant of the continuation of $h_{ij}(z)$) [5],

$$(\chi, Hf(z)) = z(\chi, f(z)) \tag{30}$$

to the pole positions, however, has solutions. On these points (the complex pole solutions of (24),(25)), $f(z)$ must be chosen so that the analytic continuation of the vectors $\{(\phi_j|f(z))\}$ are eigenfunctions with zero eigenvalue of the analytically continued matrix $h_{ij}(z)$.

We now investigate the existence of wave operators for the generalized Pietenpol system. Let us consider ${\phi_k}^{\perp}$, the set of elements in the Hilbert space orthogonal to the eigenvectors of H_0. For a set of χ dense in ${\phi_k}^{\perp}$, the existence of the wave operators requires

$$\|V\mathrm{e}^{-\mathrm{i}H_0t}\chi\| \to 0 \tag{31}$$

for $t \to \pm\infty$. Now, since χ has no component in ϕ_k,

$$\|V\mathrm{e}^{-\mathrm{i}H_0t}\chi\|^2 = \sum_{i=1}^{N} |(\phi_i, V\mathrm{e}^{-\mathrm{i}H_0t}\chi)|^2 = \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} g_i^*(\lambda)\mathrm{e}^{-\mathrm{i}\lambda t}(\lambda|\chi)\,\mathrm{d}\lambda \right|^2 . \tag{32}$$

For the set $\{\chi\}$ such that $(\lambda|\chi)$ are continuous and dense in ${\phi_k}^{\perp}$, it follows from the Riemann–Lebesgue lemma that each term of the sum in (32) vanishes for $t \to \infty$. Since the sum exists for $t = 0$ and χ in the domain of V, and each (positive) term decreases with t for t sufficiently large, (31) is valid for any finite N. For $N \to \infty$, however, the sum (31) may diverge for finite t if V is unbounded, and hence the wave operator may not exist. This can happen if the corresponding infinite rank potential approximates a local potential which is sufficiently singular or has sufficiently slow decrease at infinity.

Returning to the integral (5), we see that we may deform the path of integration C to the lower half plane, obtaining the residual contributions of the poles of the matrix $R_{ij}(z)$ successively. Since, for $|z| \to \infty$, $h_{ij}(z) \to (z - m_i)\delta_{ij}$, $R_{ij}(z) \sim \mathrm{O}(1/z)$. The integrals along the edges of the rectangle formed by the deformation $C \to C'$, a line from right to left in the lower half plane, therefore do not contribute. The contour C' can be moved below as many of the poles as the domain of analyticity of $W_{ij}(z)$ permits†. In fact, the contour C' need not be a horizontal line, but is only required to enclose the poles from below. To bound the contribution of the integral along this line, however, we may choose sufficient analyticity of $W_{ij}(z)$ to admit a contour of constant imaginary part. In this way, for N finite, we may choose a contour for which the time evolution (5) is displayed as a sum over pole contributions (exact exponential decay) plus a non-exponential 'background' which, for any $t > 0$, is bounded as $\mathrm{O}(\mathrm{e}^{-td})$, where d is determined by the domain of analyticity of $W_{ij}(z)$. For $t = 0$, this background term cannot be neglected, since (if H is defined on ψ)

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} |(\phi_i, \mathrm{e}^{-\mathrm{i}Ht}\phi_j)|^2 \bigg|_{t=0} = 0$$

i.e. exact exponential behaviour is not valid at $t = 0$.

For $N \to \infty$, unless the imaginary part of the sequence of poles is bounded, the background term corresponding to the C' integration in the finite lower half plane will carry some exponential contributions; these can be made explicit by lowering the contour. These contributions are, however, bounded by $\mathrm{O}(\mathrm{e}^{-td})$ for $t > 0$ if $W_{ij}(\zeta)$ has sufficient domain of analyticity and $W_{ij}(\zeta)/(z - \zeta)$ is integrable along $\mathrm{Im}\,\zeta = -d$.

In constructing a model of Stark-like phenomena, we note that for a potential (without Stark field) producing a discrete spectrum with ionization bound, one expects the corresponding pole positions to move further from the real axis (less stable) as the ionization point is approached. In this case, there would be no condensation of the imaginary parts

† This technique may be used whether or not V is trace class. It therefore extends the known facts about trace class operators. The t-dependence is well defined even if it is not trace class, which may be helpful in the discussion of the Stark effect in cases for which the wave operators do not exist.

even if the unperturbed levels condense on the real line. The approximation (24) (in fact valid only for small imaginary part) would imply that the g_i increase. If this implication is realized, the wave operator may not exist in the usual sense.

We finally remark that if a Hamiltonian system with continuous spectrum in $(0, \infty)$ is embedded in a Liouville space (for which the states are the Hilbert–Schmidt operators on the original Hilbert space), one finds a model of the type we have considered (the discrete spectrum of H_0 is mapped onto a discrete difference spectrum, and the continuum on $(0, \infty)$ to a continuum on $(-\infty, \infty)$), as will be discussed in a future work. It is still true that the background term must contribute at $t = 0$, so that exact exponential dependence, although possibly a very good approximation for $t > 1/d$, cannot be achieved for all t. In the same way as for the models in the usual Hilbert space [5], exact exponential behaviour can be achieved by studying the time dependence of the generalized states of the rigged Liouville space. In this case, these states are not represented by factorizable Hilbert–Schmidt operators, but are defined as linear functionals under analytic continuation as elements in a suitable extension of the the Hilbert–Schmidt space [6].

Acknowledgments

One of us (LPH) wishes to thank Professor S Adler for his hospitality at the Institute for Advanced Study, where his work was supported in part by the Monell Foundation and the National Science Foundation Grant NSF PHY91-06210. He is also grateful to Professor I Prigogine for his hospitality at the Solvay Institute at the Free University of Brussels, and the authors wish to thank him and the members of his very stimulating group for many helpful discussions.

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